

THE MONIC LAGUERRE POLYNOMIALS PRESERVE REAL-ROOTEDNESS

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ABSTRACT. Let $L_n(x)$ and $L_n^\alpha(x)$ be the n th Laguerre and associated Laguerre polynomial respectively. Fisk proved that the linear operator sending x^n to $L_n(x)$ preserves real-rootedness. In this note we prove a stronger result; namely, that when $\alpha \geq 0$, the linear operator sending x^n to $(-1)^n n! L_n^\alpha(x)$ preserves real-rootedness.

Let α be a nonnegative real number. The classical Laguerre polynomials are defined as follows:

$$L_n^\alpha(x) := \sum_{i=0}^n (-1)^i \binom{n+\alpha}{n-i} \frac{x^i}{i!}$$

Thus $(-1)^n n! L_n$ is monic for every n . Likewise, we define the *scaled Hermite polynomials* as follows:

$$H_n^\xi(x) = e^{-\xi \frac{d^2}{dx^2}} x^n$$

In this note we prove the following:

Theorem 1. *If $N > 0$ and $P(x) = \sum_{i=0}^N a_i x^i$ has all real roots, so does the polynomial $\sum_{i=0}^N (-1)^i i! a_i L_i(x)$.*

We first notice that:

$$(-1)^n n! L_n^\alpha(x) = \exp\left(-x \frac{d^2}{dx^2} - (\alpha+1) \frac{d}{dx}\right) x^n$$

To prove this, we define $\Lambda_x := x \frac{d^2}{dx^2} + (\alpha+1) \frac{d}{dx}$ and note that $\Lambda_x x^k = k(k+\alpha)x^{k-1}$ for any integer k . Thus:

$$\begin{aligned} \exp\left(x \frac{d^2}{dx^2} + (\alpha+1) \frac{d}{dx}\right) x^n &= \sum_{j=0}^{\infty} (-1)^j \frac{\Lambda_x^j}{j!} x^n \\ &= \sum_{j=0}^n \frac{(-1)^j}{j!} \left(\prod_{i=0}^{j-1} (n-i)(n+\alpha-i) \right) x^{n-j} \\ &= \sum_{j=0}^n \frac{(-1)^j n! (n+\alpha)!}{j! (n-j)! (n+\alpha-j)!} x^{n-j} \\ &= n! \sum_{i=0}^n (-1)^i \binom{n+\alpha}{i} \frac{x^{n-i}}{(n-i)!} = (-1)^n n! L_n^\alpha(x) \end{aligned}$$

The proof of Theorem 1 rests on a few important facts. First, the following properties are easily verified, when $\xi > 0, \alpha \geq 0$, and $k, \ell, m, n \in \mathbb{Z}_{\geq 0}$:

$$\int_{-\infty}^{\infty} H_k^\xi(x) H_\ell^\xi(x) e^{-x^2/4\xi} dx = 2k! \sqrt{\pi\xi} \delta_{k,\ell}$$

$$\int_0^\infty x^\alpha e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) dx = \frac{\Gamma(n + \alpha + 1)}{n!} \delta_{n,m}$$

It follows from the theory of orthogonal polynomials that both H_k^ξ and L_k^α have k distinct real roots, when $k > 0$. We also need the following limit theorems, which are new as far as we can tell:

Lemma 1. *Suppose $\xi \neq 0$, $k \in \mathbb{Z}_{>0}$, and p is a polynomial so that $p(0) \neq 0$. Let $r_k(\xi)$ be the magnitude of the largest root of H_k^ξ . Then there is a positive real $h_0 = h_0(\xi, k, p, \alpha)$ such that when $0 < h < h_0$, $e^{-h\Lambda_x}((x - \xi)^k p(x - \xi))$ has at least k distinct roots in the interval $(\xi - 2r_k(\xi)\sqrt{h}, \xi + 2r_k(\xi)\sqrt{h})$.*

Proof. Let $h = \eta^2$ with η nonnegative. Let $P_h(x) = e^{-h\Lambda_x}((x - \xi)^k p(x - \xi))$. Then we have:

$$\begin{aligned} P_h(x + \xi) &= \exp -h \left((x + \xi) \frac{d^2}{dx^2} + (\alpha + 1) \frac{d}{dx} \right) (x^k p(x)) \\ &= \exp \left(-h\xi \frac{d^2}{dx^2} - hx \frac{d^2}{dx^2} - h(\alpha + 1) \frac{d}{dx} \right) (x^k p(x)) \end{aligned}$$

Thus:

$$\begin{aligned} \frac{P_h(\epsilon\eta + \xi)}{\eta^k} &= \exp \left(-\xi \frac{d^2}{d\epsilon^2} - \eta\epsilon \frac{d^2}{d\epsilon^2} - \eta(\alpha + 1) \frac{d}{d\epsilon} \right) (\epsilon^k p(\epsilon\eta)) \\ &= \exp \left(-\xi \frac{d^2}{d\epsilon^2} - \eta\Lambda_\epsilon \right) (\epsilon^k p(\epsilon\eta)) \end{aligned}$$

This latter function is a polynomial in η of degree k whose coefficients are polynomials in ϵ , ξ and α independent of η . Hence, if $|\epsilon| < 2r_k(\xi)$, there is a positive η_0 such that for $|\eta| < \eta_0$,

$$\left| \frac{P_h(\epsilon\eta + \xi)}{\eta^k} - p(0)H_k^\xi(\epsilon) \right| \leq C\eta$$

for some constant C . Also note that the scaled Hermite polynomial H_k^ξ has k distinct real roots in the interval $(-2r_k(\xi), 2r_k(\xi))$, and thus there are $k + 1$ numbers $a_1 < a_2 < \dots < a_{k+1}$ in that interval so that the signs of $H_k^\xi(a_j)$ alternate, for $j = 1, 2, \dots, k + 1$. Thus for small η , the signs of $P_h(a_j\eta + \xi)$ alternate as well, so by the intermediate value theorem, $P_h(x)$ has at least k roots in the interval $(\xi - 2r_k(\xi)\sqrt{h}, \xi + 2r_k(\xi)\sqrt{h})$. This proves the lemma. \square

Lemma 2. *Suppose $k \in \mathbb{Z}_{>0}$, and p is a polynomial so that $p(0) \neq 0$. Let $s_k(\alpha)$ be the magnitude of the largest root of L_k^α . Then there is a positive real $h_0 = h_0(k, p, \alpha)$ such that when $0 < h < h_0$, $e^{-h\Lambda_x}(x^k p(x))$ has at least k distinct roots in the interval $(-2s_k(\alpha)h, 2s_k(\alpha)h)$.*

Proof. The proof is almost identical to that of Lemma 1. Let $P_h(x) = e^{-h\Lambda_x}(x^k p(x))$. Then we have:

$$\begin{aligned} \frac{P_h(\epsilon h)}{h^k} &= \exp \left(-\epsilon \frac{d^2}{d\epsilon^2} - (\alpha + 1) \frac{d}{d\epsilon} \right) (\epsilon^k p(\epsilon h)) \\ &= \exp(-\Lambda_\epsilon) (\epsilon^k p(\epsilon h)) \end{aligned}$$

This latter function is a polynomial in h of degree k whose coefficients are polynomials in ϵ and α independent of h . Hence, if $|\epsilon| < 2s_k(\alpha)$, there is a positive h_* such that for $|h| < h_*$,

$$\left| \frac{P_h(\epsilon h)}{h^k} - p(0)L_k^\alpha(\epsilon) \right| \leq Ch$$

for some constant C . Also note that $L_k^\alpha(\epsilon)$ has k distinct real roots in the interval $(-2s_k(\alpha), 2s_k(\alpha))$, and thus there are $k+1$ numbers $a_1 < a_2 < \dots < a_{k+1}$ in that interval so that the signs of $L_k^\alpha(a_j)$ alternate, for $j = 1, 2, \dots, k+1$. Thus for small h , the signs of $P_h(a_j h)$ alternate as well, so by the intermediate value theorem, $P_h(x)$ has at least k roots in the interval $(-2s_k(\alpha)h, 2s_k(\alpha)h)$. This proves the lemma. \square

Now we are ready to prove Theorem 1. Fix a non-constant real-rooted polynomial $P(x) = \sum_{i=0}^N a_i x^i$, and suppose that

$$P(x) = \prod_{i=1}^n (x - \xi_i)^{m_i}$$

where $m_0, m_1, \dots, m_n \in \mathbb{Z}_{>0}$, and $\xi_1 < \xi_2 < \dots < \xi_n$ are real numbers.

Lemmas 1 and 2 imply that for any i , there exists h_i so that for all $h \in (0, h_i)$, the polynomial $e^{-h\Lambda_x} P(x)$ has m_i roots in the interval $I_i(h) = (\xi_i - 2r_k(\xi_i)\sqrt{h}, \xi_i + 2r_k(\xi_i)\sqrt{h})$ if $\xi_i \neq 0$, and $I_i(h) = (-2s_k(\alpha)h, 2s_k(\alpha)h)$ if $\xi_i = 0$. Choose a positive real number h' so that:

- (1) $0 < h' < h_i$ for all i , and
- (2) the intervals $I_i(h')$ are disjoint.

Then for every $h \in (0, h')$, $e^{-h\Lambda_x} P(x)$ has at least $m := m_1 + m_2 + \dots + m_n$ distinct roots. Since the degree of $e^{-h\Lambda_x} P(x)$ is that of $P(x)$, namely m , it follows that $e^{-h\Lambda_x} P(x)$ has m real roots with multiplicity 1.

We now define the set:

$$H(P) = \{h' \in (0, \infty) \mid e^{-h\Lambda_x} P(x) \text{ has } m \text{ simple real roots for all } h \in (0, h')\}$$

Then the above argument shows that $H(P)$ is nonempty and open in $(0, \infty)$. In addition, it is clear from the definition that $H(P)$ is either $(0, \infty)$ or an interval of the form $(0, y)$ for some positive real y . If the latter were true, then $e^{-h\Lambda_x} P(x)$ would have m simple real roots for all $h \in (0, h')$, for all $h' < y$. However, since $\cup_{h' < y} (0, h') = (0, y)$, it follows that $y \in H(P)$, a contradiction. Hence $H(P) = (0, \infty)$. However, we know that $P(x) = \sum_{i=0}^N a_i x^i$, so that

$$e^{-\Lambda_x} P(x) = \sum_{i=0}^N (-1)^i i! a_i L_i(x)$$

Since $1 \in H(P)$, it follows that the latter polynomial has N distinct real roots, and Theorem 1 is proven.

BIBLIOGRAPHY

1. Fisk, Steve. *The Laguerre polynomials preserve real-rootedness*. [\protect\vrule width0pt\protect\href{}](#)
2. Szwarc, Ryszard. *Orthogonal Polynomials and Banach Algebras*. In *Inzell Lectures on Orthogonal Polynomials*, edited by Wolfgang zu Castell, Frank Filbir, and Brigitte Forster.